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A proof of the generalized Nakayama conjecture for algebras with $J^{2l+1}=0$ and A/J^l representation finite

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The classical Nakayama conjecture states that a finite-dimensional algebra A over a field k is injective provided $\operatorname{domdim}_A A=\infty$, that is, in a minimal injective resolution of the left A-module A each of the appearing injective modules is also projective (see [5, 4]). The validity of several other conjectures, for example, the generalized Nakayama conjecture (see [1]) or the conjecture on the finiteness of the finitistic dimension (see [2]), would imply that the classical Nakayama conjecture holds.

The (right) generalized Nakayama conjecture says that in the minimal injective resolution of the right A-module A every indecomposable injective right A-module appears as direct summand. Using the usual duality $D:=\operatorname{Hom}_k(-,k)$ this is equivalent to the appearance of every indecomposable projective left A-module in the minimal projective resolution of the left module DA. This again can be reformulated to the assertion that for each simple left A-module S there exists a nonnegative integer i such that $\operatorname{Ext}^i_A(DA,S)\neq 0$.

The even more general conjecture on the finiteness of the finitistic dimension was proved recently for algebras (even for artinian rings) whose Jacobson radical J satisfies $J^3=0$ (see [3]). We had the feeling that this result could be generalized to algebras with $J^{2l+1}=0$ and A/J^l representation finite. Of course, $J^3=0$ would then be a special case for $l=1$. As we could not adapt the proof of [3], we started to work out a proof of our own which is still somehow in the spirit of [3] because it reduces the question to a triviality about finite-dimensional vector spaces.

Unfortunately our technique up to now is not strong enough to prove that the finitistic dimensions of these algebras are finite. Nevertheless, it provides a (as we think short and easy) proof of the generalized Nakayama conjecture for these algebras. In fact, in a previous version of this paper the first author could only prove the classical Nakayama conjecture. The second author then pointed out how the generalized Nakayama conjecture can be attacked in a similar fashion. So the aim of this note is to prove the following result:

Theorem. *Let A be a finite-dimensional algebra over a field k with radical J. If there is a natural number l such that $J^{2l+1}=0$ and A/J^l is representation finite, then the generalized Nakayama conjecture is true for A.*

Proof. We give a proof by contradiction and assume that there is an algebra A satisfying our assumptions such that there is a simple left A-module S with the property $\mathrm{Ext}_A^i(DA, S)=0$ for all $i \geq 0$.

Considering the injective hull Q_0 of S we denote by E the largest submodule of Q_0 such that each composition factor of E is isomorphic to S. In particular, $\mathrm{Hom}_A(S, Q_0/E)=0$ and $\mathrm{Ext}_A^i(DA, E)=0$ for all $i \geq 0$. Hence the minimal injective resolution

$$0 \to E \to Q_0 \to Q_1 \to Q_2 \to \cdots$$

of E is mapped to an exact sequence

$$0 \to \nu^- Q_0 \xrightarrow{\alpha_0} \nu^- Q_1 \xrightarrow{\alpha_1} \nu^- Q_2 \xrightarrow{\alpha_2} \cdots$$

by the inverse Nakayama functor $\nu^- = \mathrm{Hom}_A(DA, -)$. Setting $X_i := \mathrm{Im}\, \alpha_i$ for all $i \geq 0$ we obtain from the minimality of the injective resolution of E that

$$0 \to \nu^- Q_0 \xrightarrow{\alpha_0} \nu^- Q_1 \xrightarrow{\alpha_1} \cdots \nu^- Q_i \to X_i \to 0$$

is a minimal projective resolution of X_i and the module X_i satisfies $J^{2l}X_i = 0$. Moreover, the choice of Q_0 and E shows $\mathrm{Hom}_A(\nu^- Q_0, S) \cong \mathrm{Hom}_A(S, Q_0) \neq 0$ and $\mathrm{Hom}_A(\nu^- Q_1, S) \cong \mathrm{Hom}_A(S, Q_1) = 0$.

That A/J^l is representation finite means that we can find a finite set $\{C_1, \ldots, C_n\}$ of representatives of all isomorphism classes of indecomposable finite-dimensional A/J^l-modules. Given now an arbitrary finite-dimensional A-module M with $J^{2l}M=0$, then in the exact sequence

$$0 \to J^l M \to M \to M/J^l M \to 0$$

the first and the last term are actually A/J^l-modules. Hence they can be written as $M_1 := J^l M \cong \bigoplus_{j=1}^n C_j^{s_j}$ and $M_2 := M/J^l M \cong \bigoplus_{j=1}^n C_j^{t_j}$.

If we assume pdim $M \leq p$ for some $p \in \mathbb{N}$, then in the long exact sequence

$$\begin{aligned}\cdots \to \operatorname{Ext}_A^i(M_2, S) &\to \operatorname{Ext}_A^i(M, S) \to \operatorname{Ext}_A^i(M_1, S)\\ &\to \operatorname{Ext}_A^{i+1}(M_2, S) \to \operatorname{Ext}_A^{i+1}(M, S) \to \cdots\end{aligned}$$

the term $\operatorname{Ext}_A^i(M, S)$ vanishes for all $i > p$. Consequently for all $i > p$ the k-vectorspaces $\operatorname{Ext}_A^i(M_1, S)$ and $\operatorname{Ext}_A^{i+1}(M_2, S)$ are isomorphic and so their k-dimensions are equal. This means that the vector $(s_1, \ldots, s_n, t_1, \ldots, t_n) \in \mathbb{Q}^{2n}$ satisfies the equations $\sum_{j=1}^n a(i,j)s_j = \sum_{j=1}^n a(i+1,j)t_j$ for all $i > p$ if we set $a(i,j) := \dim_k \operatorname{Ext}_A^i(C_j, S)$. In other words, we have shown $(s_1, \ldots, s_n, t_1, \ldots, t_n) \in L_p$ if we define L_p as the $\mathbb{Q}$-subspace of $\mathbb{Q}^{2n}$ given by all vectors $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that $\sum_{j=1}^n a(i,j)x_j = \sum_{j=1}^n a(i+1,j)y_j$ for all $i > p$.

As $(L_p)_{p \in \mathbb{N}}$ is an ascending chain of subspaces of $\mathbb{Q}^{2n}$, there exists an index $p_0 \in \mathbb{N}$ with $L_p \subseteq L_{p_0}$ for all $p \in \mathbb{N}$. Choosing now $M := X_{p_0+2}$ we have $(s,t) \in L_{p_0+2} \subseteq L_{p_0}$. This means that in the exact sequence

$$\begin{aligned}\operatorname{Ext}_A^{p_0+1}(M, S) \xrightarrow{\alpha} \operatorname{Ext}_A^{p_0+1}(M_1, S) &\to \operatorname{Ext}_A^{p_0+2}(M_2, S)\\ &\to \operatorname{Ext}_A^{p_0+2}(M, S) \to \operatorname{Ext}_A^{p_0+2}(M_1, S)\\ &\to \operatorname{Ext}_A^{p_0+3}(M_2, S) \to \operatorname{Ext}_A^{p_0+3}(M, S) = 0\end{aligned}$$

the second and third term and also the fifth and sixth term have equal k-dimension. The inequality

$$\begin{aligned}\dim_k(\nu^- Q_0, S) &= \dim_k \operatorname{Ext}_A^{p_0+2}(M, S)\\ &\leq \dim_k \operatorname{Ext}_A^{p_0+1}(M, S)\\ &= \dim_k \operatorname{Hom}_A(\nu^- Q_1, S)\end{aligned}$$

yields the contradiction $\operatorname{Hom}_A(\nu^- Q_1, S) \neq 0$. $\square$

Remark. There are other situations where we can use similar arguments as above to prove the generalized Nakayama conjecture. An example is the case that A has an idempotent e such that eAe is a skew field and A/AeA is representation finite. Then for each A-module M the factor module $M_2 := M/AeM$ and the kernel M_1' of the projective cover of the submodule $M_1 := AeM$ are both actually A/AeA-modules. As $\operatorname{Ext}_A^i(M_1, -) \cong \operatorname{Ext}_A^{i-1}(M_1', -)$ for all $i > 1$ and $\operatorname{Ext}_A^{i+1}(M_2, -) \cong \operatorname{Ext}_A^i(M_1, -)$ if pdim $M < i$ we can apply our vectorspace argument using the decompositions of M_1' and M_2 into indecomposable A/AeA-modules.

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